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Results in Mathematics

A Singular Behaviour of a Set-Valued Approximate Orthogonal Additivity

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Dedicated to Professor Zygfryd Kominek on the occasion of his 70th birthday

Abstract. We show that unlikely to the single-valued case, the set-valued orthogonally additive equation is unstable. After presenting an example showing this phenomenon, we provide some special cases where a set-valued approximately orthogonally additive function can be approximated by the one which satisfies the equation of orthogonal additivity exactly.

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1. Introduction

We call function $f\colon X \to Y$ orthogonally additive if it satisfies the conditional functional equation

$$f(x+y) = f(x) + f(y) \quad \text{for all } \ x, y \in X \quad \text{with } \ x \perp y. \tag{1}$$

In the standard settings X is a real inner product space with the orthogonality relation given by means of the inner product and Y is an Abelian group. However, we may introduce an abstract orthogonality relation in any at least two-dimensional linear space, defining the so called orthogonality space (see Gudder and Strawther [7], Rätz [12]).

Let X be a real linear space with $\dim X \geq 2$ and let $\perp$ be a binary relation on X such that

(01) $x \perp 0$ and $0 \perp x$ for all $x \in X$;

(02) if $x, y \in X\setminus\{0\}$ and $x \perp y$, then x and y are linearly independent;
(03) if $x, y \in X$ and $x \perp y$, then for all $\alpha, \beta \in \mathbb{R}$ we have $\alpha x \perp \beta y$;
(04) for any two-dimensional subspace P of X, for every $x \in P$ and for every $\lambda > 0$ there exists a $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

An ordered pair $(X, \perp)$ is called an *orthogonality space*.

An orthogonality space covers the case of an inner product space with the classical orthogonality as well as an arbitrary real normed linear space with the so called Birkhoff orthogonality. But it is also the case with the "trivial" orthogonality defined on a linear space by (01) and the condition that two nonzero vectors are orthogonal if and only if they are linearly independent.

Solutions of (1) are known (see Rätz [12], and also Baron and Volkmann [2]). Before giving their form, we recall that a function q is called quadratic if for all x and y from the domain, $q(x + y) + q(x - y) = 2q(x) + 2q(y)$.

Theorem 1.1. *Let $(X, \perp)$ be an orthogonality space and $(Y, +)$ be an Abelian group. Every orthogonally additive function $f : X \to Y$ has the form $f = a+q$, where a is additive and q is quadratic (and orthogonally additive).*

The orthogonal additivity (1) has wide applications both inside and outside mathematics. With help of it we can give, e.g., several characterizations of inner product spaces among normed spaces as well as of Hilbert spaces among Banach spaces (see Rätz [13] or Sikorska [16] for more reference items). Equation (1) has its applications in physics, in the theory of ideal gas (see, e.g., Aczél and Dhombres [1], Truesdell and Muncaster [18]). In the three-dimensional Euclidean space, by means of (1) we obtain the formula for the distribution law of velocities in an ideal gas at a fixed temperature. There are also applications of (1) in actuarial mathematics in a premium calculation principle (see Heijnen and Goovaerts [9]): it is shown, namely, that the variance principle is the only covariance-additive premium principle.

The multi-valued analogue of (1) reads as follows

$$F(x + y) = F(x) + F(y) \quad \text{for all } x, y \in X \quad \text{with } x \perp y, \tag{2}$$

where F maps an orthogonality space X into the family of non-empty subsets of a topological space Y (for nonempty sets A and B, by $A+B$ we understand $\{a + b : a \in A, b \in B\}, \lambda A = \{\lambda a : a \in A\}$ for any $\lambda \in \mathbb{R}$ and $A - B = A + (-1)B$).

In what follows we will use the notations: for a metric linear space Y, let $c(Y)$ denote the family of all nonempty compact subsets of $Y, cc(Y)$—the family of all convex members of $c(Y)$, and $bcl(Y)$—the collection of all nonempty, closed and bounded subsets of Y.

We know the solutions of (2) (see Sikorska [17]).

Theorem 1.2. *Let $(X, \perp)$ be an orthogonality space and Y be a Fréchet space (locally convex, complete metric linear space with an invariant metric). If $F : X \to cc(Y)$ satisfies (2), then there exist an additive function $a : X \to Y$*

and a quadratic function $Q\colon X \to cc(Y)$ *such that* $F = a + Q$. *Moreover, such representation is unique.*

The main aim of this paper is to study approximately orthogonally additive multi-valued functions. We will answer the question whether for an approximately orthogonally additive function there exists a function which satisfies the conditional Cauchy equation exactly and which is close (in the sense of the Hausdorff distance) to the given function. In such a case we tell that a (multi-valued) orthogonal Cauchy equation is stable.

Let us recall the stability result for the single-valued orthogonal Cauchy equation (cf., Fechner and Sikorska [5], Ger and Sikorska [6]).

Theorem 1.3. *Let* X *be an orthogonality space and let* $(Y, \|\cdot\|)$ *be a (real or complex) Banach space. Given an* $\varepsilon \geq 0$, *let* $f\colon X \to Y$ *be a mapping such that for all* $x, y \in X$ *one has*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \quad \text{for all } x, y \in X \quad \text{with } x \perp y. \tag{3}$$

Then there exists a mapping $g\colon X \to Y$ *such that*

$$g(x+y) = g(x) + g(y) \quad \text{for all } x, y \in X \quad \text{with } x \perp y, \tag{4}$$

and

$$\|f(x) - g(x)\| \leq 5\varepsilon \quad \text{for all } x \in X. \tag{5}$$

Moreover, such mapping g, *for which the difference* $f - g$ *is bounded, is unique and it is given by the formula*

$$g(x) = \lim_{n\to\infty} \left(\frac{2^n + 1}{2 \cdot 4^n} f(2^n x) - \frac{2^n - 1}{2 \cdot 4^n} f(-2^n x) \right) \quad \text{for all } x \in X.$$

In what follows we present some background for further considerations.

Lemma 1.1. (Rådström [11]) *Assume that* A, B, C *are subsets of a normed linear space* Y *such that* B *is closed and convex,* C *is bounded, nonempty, and* $A + C \subset B + C$. *Then* $A \subset B$.

Let $(Y, \|\cdot\|)$ be a normed linear space. On the set of all nonempty closed and bounded subsets of Y we define the distance function, which is called the *Hausdorff distance*, as follows. For any $A, B \in bcl(Y)$,

$$h(A, B) := \max \left\{ \sup_{x\in A} d(x, B), \sup_{y\in B} d(A, y) \right\},$$

where $d(x, B) = d(B, x) := \inf \big\{\|x - y\|\colon y \in B\big\}$. Or, equivalently,

$$h(A, B) = \inf \{\varepsilon > 0\colon A \subset B + K(\varepsilon) \text{ and } \subset A + K(\varepsilon)\},$$

where $K(\varepsilon)$ is simply a closed ball of radius ε centered at the origin. Surely, we have $K(\varepsilon_1) + K(\varepsilon_2) = K(\varepsilon_1 + \varepsilon_2)$ and $K(\alpha\varepsilon) = \alpha K(\varepsilon)$ with any $\alpha > 0$.

Space $bcl(Y)$ with the Hausdorff distance forms a metric space.

It is known that if Y is complete, so is $cc(Y)$, if considered with the Hausdorff metric (see, e.g., Castaing and Valadier [4, Chapter II] or Beer [3, Section 3.2]).

Lemma 1.2. *Assume that Y is a normed linear space and the considered sets are from the family $cc(Y)$. Then*

(i) $h(A + C, B + C) = h(A, B)$;
(ii) $h(\alpha A) = \alpha h(A)$ *for any* $\alpha > 0$;
(iii) *if* $A_n \to A$ *and* $B_n \to B$, *then* $A_n + B_n \to A + B$;
(iv) *if* $A_n \to A$ *and* $B_n \to B$ *then* $h(A_n, B_n) \to h(A, B)$.

2. Main Results

Let $(X, \perp)$ be an orthogonality space. In what follows we will study the condition

$$h(F(x + y), F(x) + F(y)) \leq \varepsilon \quad \text{for all } x, y \in X \quad \text{with } x \perp y. \tag{6}$$

Mirmostafaee and Mahdavi [10] were studying stability of a set-valued version of the equation of orthogonal additivity but only for even functions. They have obtained a stability result in such a case. However, the problem appears in the case where we consider odd mappings, or just arbitrary set-valued functions. It seems to be a challenge. We may not treat separately the odd and even parts of a function. Indeed, if for every $x \in X$ we denote

$$G(x) := \frac{1}{2}(F(x) + F(-x)) \quad \text{and} \quad H(x) := \frac{1}{2}(F(x) - F(-x)),$$

then in general,

$$F(x) \neq G(x) + H(x).$$

We start with a lemma, which we prove using (6) and properties of the orthogonality relation (see [17, Theorem 2.1] for all ten suitable orthogonality relations between respective vectors), using properties of the Hausdorff distance, and finally applying Lemma 1.1.

Lemma 2.1. *Let $(X, \perp)$ be an orthogonality space and $(Y, \|\cdot\|)$ be a real normed space. If $F \colon X \to cc(Y)$ satisfies (6), then for every $n \in \mathbb{N}$,*

$$h\left(\frac{2^n + 1}{2 \cdot 4^n} F(2^n x), F(x) + \frac{2^n - 1}{2 \cdot 4^n} F(-2^n x)\right) \leq \left(5 - \frac{5}{2^n}\right)\varepsilon \quad \text{for all } x \in X. \tag{7}$$

In an arbitrary normed space Y by use of (7) we are not able to define any Cauchy sequence (as it is done in a standard procedure of a direct method while proving the stability; see, e.g., [8]). In order to see a possible behaviour of approximately set-valued orthogonally additive functions we restrict ourselves to the case $Y = \mathbb{R}$ and we will consider the family $cc(\mathbb{R})$ consisting of all nonempty closed and bounded intervals.

In what follows we will show that it is not caused only by a method of a proof that by means of Lemma 2.1 we could not build an orthogonally additive approximation. Namely, we will give an example showing that unlikely to the single-valued case, the set-valued orthogonally additive equation is in general unstable. However, later on we present some special cases where a set-valued approximately orthogonally additive function can be approximated by the one which satisfies the equation of orthogonal additivity exactly.

For two arbitrary sets $A = [a_1, a_2]$ and $B = [b_1, b_2]$ from $cc(\mathbb{R})$ we have now

$$h(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}.$$

In our settings function $F\colon X \to cc(\mathbb{R})$ has now the form

$$F(x) = [a(x), b(x)] \quad \text{for all } \ x \in X \tag{8}$$

with some functions $a, b\colon X \to \mathbb{R}, a \le b$, and the fact that F satisfies (6) is then equivalent to the fact that a and b satisfy system of conditions

$$\begin{cases} \big|a(x+y) - a(x) - a(y)\big| \le \varepsilon & \text{for all } \ x, y \in X \quad \text{with } \ x \perp y \\ \big|b(x+y) - b(x) - b(y)\big| \le \varepsilon & \text{for all } \ x, y \in X \quad \text{with } \ x \perp y. \end{cases} \tag{9}$$

Before proceeding with stability considerations we formulate two simple lemmas concerning orthogonally additive functions.

Lemma 2.2. *Function $F\colon X \to cc(\mathbb{R})$ of the form (8) is orthogonally additive if and only if $a\colon X \to \mathbb{R}$ and $b\colon X \to \mathbb{R}$ are orthogonally additive and $a \le b$.*

Lemma 2.3. *If $a, b\colon X \to \mathbb{R}$ are orthogonally additive and $a \le b$ then odd parts of these functions coincide.*

Proof. If a and b are orthogonally additive, so is $d := b - a \ge 0$. And since d is orthogonally additive, $d = h + q$ with some additive h and quadratic q. Then for every $x \in X$ and $n \in \mathbb{N}$ we have

$$h(x) + \frac{1}{2^n} q(x) = 2^n h\left(2^{-n}x\right) + 2^n q\left(2^{-n}x\right) = 2^n d\left(2^{-n}x\right) \ge 0,$$

that is, $h \ge 0$ and consequently, the odd part of d, which is uniquely determined, is equal to zero. □

Remark 2.1. The result of the above lemma can also be shown by means of Theorem 1.2, however we have done it directly.

We present now the announced already example.

Example 2.1. Consider Euclidean space $\mathbb{R}^2$, some $\varepsilon > 0$ and $F\colon \mathbb{R}^2 \to cc(\mathbb{R})$ of the form (8) with $a(x_1, x_2) = \frac{1}{2}\varepsilon\left(x_1^2 + x_2^2 + x_1 + 1\right) \ge 0$ and $b(x_1, x_2) = \varepsilon\left(x_1^2 + x_2^2 + x_1 + 1\right) \ge 0$. Then $a \le b$ and functions a and b satisfy system (9). Consequently, F satisfies (6).

If F were stable, there would exist an orthogonally additive function $G\colon \mathbb{R}^2 \to cc(\mathbb{R})$ such that

$$h\big(F(x), G(x)\big) \le M \quad \text{for all } \ x=(x_1,x_2)\in\mathbb{R}^2$$

with some constant M (depending on ε).

Since G is orthogonally additive, by Theorem 1.2, $G = g + Q$ with $g\colon \mathbb{R}^2 \to \mathbb{R}$ additive and $Q\colon \mathbb{R}^2 \to cc(\mathbb{R})$ quadratic. Then $Q(x) = [q_1(x), q_2(x)]$ with some quadratic functions $q_1, q_2\colon \mathbb{R}^2 \to \mathbb{R}, q_1 \le q_2$, and $G(x) = [g(x) + q_1(x), g(x) + q_2(x)]$ for all $x \in \mathbb{R}^2$.

It follows from (9) and Theorem 1.3 that there exist orthogonally additive functions $\tilde{a}, \tilde{b}\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$|a(x) - \tilde{a}(x)| \le 5\varepsilon \quad \text{and} \quad |b(x) - \tilde{b}(x)| \le 5\varepsilon \quad \text{for all } \ x \in \mathbb{R}^2.$$

It is not difficult to verify that in our case $\tilde{a}(x_1,x_2) = \frac{1}{2}\varepsilon\left(x_1^2 + x_2^2 + x_1\right)$ and $\tilde{b}(x_1,x_2) = \varepsilon\left(x_1^2 + x_2^2 + x_1\right)$.

Since

$$h\big(F(x), G(x)\big) = \max\left\{\big|a(x) - g(x) - q_1(x)\big|, \big|b(x) - g(x) - q_2(x)\big|\right\}$$

is bounded, by the uniqueness of $\tilde{a}$ and $\tilde{b}$, it follows that

$$\tilde{a}(x) = g(x) + q_1(x) \quad \text{and} \quad \tilde{b}(x) = g(x) + q_2(x) \quad \text{for all } \ x \in \mathbb{R}^2.$$

This is impossible since odd parts of $\tilde{a}$ and $\tilde{b}$ equal $\frac{1}{2}\varepsilon x_1$ and εx_1, respectively, are different (cf., Lemma 2.3).

We give now two stability results obtained in two particular situations.

Theorem 2.1. *Let $(X, \perp)$ be an orthogonality space and let $F\colon X \to cc(\mathbb{R})$ of the form (8) satisfy (6) with some $\varepsilon > 0$. If $b(x) - a(x) > 10\varepsilon$ for all $x \in X\backslash\{0\}$, then there exists a unique orthogonally additive function $G\colon X \to cc(\mathbb{R})$ such that*

$$h\big(F(x), G(x)\big) \le 5\varepsilon \quad \textit{for all } \ x \in X.$$

Proof. Functions a and b satisfy (9). By Theorem 1.3, there exist orthogonally additive functions $\tilde{a}$, $\tilde{b}\colon X \to \mathbb{R}$ such that

$$\big|a(x) - \tilde{a}(x)\big| \le 5\varepsilon \quad \text{and} \quad \big|b(x) - \tilde{b}(x)\big| \le 5\varepsilon \quad \text{for all } \ x \in X. \tag{10}$$

We have

$$\tilde{a}(x) \le a(x) + 5\varepsilon \le b(x) - 5\varepsilon \le \tilde{b}(x) \quad \text{for all } \ x \in X\backslash\{0\}.$$

Since $\tilde{a}(0) = \tilde{b}(0) = 0$, we have

$$\tilde{a}(x) \le \tilde{b}(x) \quad \text{for all } \ x \in X.$$

On account of Lemma 2.2, function $G\colon X \to cc(\mathbb{R})$ defined by

$$G(x) := \big[\tilde{a}(x), \tilde{b}(x)\big] \quad \text{for all } \ x \in X$$

is orthogonally additive. Moreover,

$$h\big(G(x), F(x)\big) = \max\left\{\left|\tilde{a}(x) - a(x)\right|, \left|\tilde{b}(x) - b(x)\right|\right\} \le 5\varepsilon \quad \text{for all } x \in X.$$

In order to prove the uniqueness, let $\overline{G}$ be another function such that the (Hausdorff) distance between F and $\overline{G}$ is bounded, that is, $h\big(F(x), \overline{G}(x)\big) \le M$ for all $x \in X$ and some $M > 0$. Hence,

$$h\big(G(x), \overline{G}(x)\big) \le 5\varepsilon + M \quad \text{for all } x \in X.$$

By Theorem 1.2, $G = a_1 + Q_1$ and $\overline{G} = a_2 + Q_2$ for some additive functions $a_1, a_2 \colon X \to \mathbb{R}$ and quadratic functions $Q_1, Q_2 \colon X \to cc(\mathbb{R})$, therefore

$$h\big(a_1(x) + Q_1(x), a_2(x) + Q_2(x)\big) \le 5\varepsilon + M \quad \text{for all } x \in X,$$

and

$$h\big(a_1\left(2^n x\right) + Q_1\left(2^n x\right), a_2\left(2^n x\right) + Q_2\left(2^n x\right)\big) \le 5\varepsilon + M \quad \text{for all } x \in X,\ n \in \mathbb{N},$$
$$h\big(2^n a_1(x) + 4^n Q_1(x), 2^n a_2(x) + 4^n Q_2(x)\big) \le 5\varepsilon + M \quad \text{for all } x \in X,\ n \in \mathbb{N}.$$

Dividing by 4^n and letting n to infinity, yield $Q_1 = Q_2$. Now, it is already easy to see that $a_1 = a_2$, and the proof is completed. □

Theorem 2.2. *Let $(X, \perp)$ be an orthogonality space and let $F \colon X \to cc(\mathbb{R})$ of the form (8) satisfy (6) with some $\varepsilon \ge 0$. If $M := \sup\{|b_o(x) - a_o(x)| \colon x \in X\} < \infty$, where a_o, b_o stand for the odd parts of a and b, respectively, then there exists an orthogonally additive function $G \colon X \to cc(\mathbb{R})$ such that*

$$h(F(x), G(x)) \le 5\varepsilon \quad \textit{for all } x \in X.$$

Proof. Consider function $d := b - a$. Then $d_o = b_o - a_o$ on account of (9) satisfies

$$\big|d_o(x + y) - d_o(x) - d_o(y)\big| \le 2\varepsilon \quad \text{for all } x, y \in X \quad \text{with } x \perp y.$$

Moreover,

$$|d_o(x)| \le M \quad \text{for all } x \in X. \tag{11}$$

By Theorem 1.3, there exists an orthogonally additive function $\widetilde{d_o}$ such that

$$\left|d_o(x) - \widetilde{d_o}(x)\right| \le 10\varepsilon \quad \text{for all } x \in X. \tag{12}$$

In fact, function $\widetilde{d_o}$ is given by

$$\widetilde{d_o}(x) = \lim_{n \to \infty} \left(\frac{2^n + 1}{2 \cdot 4^n}\, d_o\left(2^n x\right) - \frac{2^n - 1}{2 \cdot 4^n}\, d_o\left(-2^n x\right)\right) \quad \text{for all } x \in X,$$

which on account of (11) yields $\widetilde{d_o} = 0$. This means that for orthogonally additive approximations of a_o and b_o, we have $\widetilde{a_o} = \widetilde{b_o}$.

Since $a(x) \le b(x)$ for all $x \in X$, the even parts a_e, b_e of a and b, respectively, also satisfy $a_e(x) \le b_e(x)$ for all $x \in X$. Hence, their orthogonally additive approximations $\widetilde{a_e}$ and $\widetilde{b_e}$ are quadratic and satisfy

$$\widetilde{a_e}(x) = \lim_{n\to\infty} \frac{1}{4^n}\, a_e(2^n x) \le \lim_{n\to\infty} \frac{1}{4^n}\, b_e(2^n x) = \widetilde{b_e}(x) \quad \text{for all } \ x \in X.$$

Therefore, function $X \ni x \mapsto Q(x) := \big[\widetilde{a_e}(x), \widetilde{b_e}(x)\big]$, is orthogonally additive and quadratic.

Define $G(x) := \{\widetilde{a_e}(x)\} + Q(x)$ for all $x \in X$. Then G is orthogonally additive and on account of the uniqueness of the orthogonally additive approximations, we have

$$h\big(F(x), G(x)\big) = h\big([a(x), b(x)], [\widetilde{a_o}(x) + \widetilde{a_e}(x), \widetilde{a_o}(x) + \widetilde{b_e}(x)]\big) \le 5\varepsilon \quad \text{for all } \ x \in X.$$

For the uniqueness of G, we use the same argument as in the proof of the previous theorem. □

Remark 2.2. By (12), the assumptions of Theorem 2.2 force $M \le 10\varepsilon$.

Consider now the following condition

$$h\big(F(x+y), F(x) + F(y)\big) \le \varphi(x, y) \quad \text{for all } \ x, y \in X \quad \text{with } \ x \perp y, \tag{13}$$

where F maps an orthogonality space $(X, \perp)$ into the family of nonempty, compact and convex subset of a real Banach space Y, and $\varphi\colon X \to [0, \infty)$ satisfies three conditions:

(a) for every $x \in X$ the series $\sum_{n=1}^{\infty} 4^n \varphi(2^{-n}x, 2^{-n}x)$ is convergent;
(b) for all $x, y \in X$ such that $x \perp y$ we have $\lim_{n\to\infty} 4^n \varphi(2^{-n}x, 2^{-n}y) = 0$;
(c) there exists an $M > 0$ such that for all $x, y \in X$, if $x \perp y$ and $x+y \perp x-y$ then

$$\max\big\{\varphi\big(\pm x, \pm y\big), \varphi\big(\pm(x+y), \pm(x-y)\big)\big\} \le M\varphi(x, x).$$

Considerations as for Lemma 2.1 lead to an approximation of the following Hausdorff distance

$$h\left(F(x), \frac{4^n + 2^n}{2}\, F\Big(\frac{x}{2^n}\Big) + \frac{4^n - 2^n}{2}\, F\Big(-\frac{x}{2^n}\Big)\right) \tag{14}$$

(see [14,15] for the single-valued case). By properties of the Hausdorff distance and of function φ we may define function $G\colon X \to cc(Y)$ by the formula

$$G(x) = \lim_{n\to\infty}\left[\frac{4^n + 2^n}{2}\, F\Big(\frac{x}{2^n}\Big) + \frac{4^n - 2^n}{2}\, F\Big(-\frac{x}{2^n}\Big)\right] \quad \text{for all } \ x \in X.$$

It turns out that G is orthogonally additive, and moreover,

$$h\big(F(x), G(x)\big) \le \psi(x) \quad \text{for all } \ x \in X,$$

with some function $\psi\colon X \to [0, \infty)$ depending on φ.

Of course, as one can easily check, function $\varphi(x) \equiv \varepsilon$ does not satisfy (a), (b). However, an example of φ that satisfies (a) and (b) is given, e.g., by $\varphi(x,y) = \varepsilon\left(\|x\|^p + \|y\|^p\right),\ (x,y) \in X^2$, with $p > 2$, $\varepsilon > 0$. For X it is enough to take then an inner product space or a normed linear space with the Birkhoff orthogonality.

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